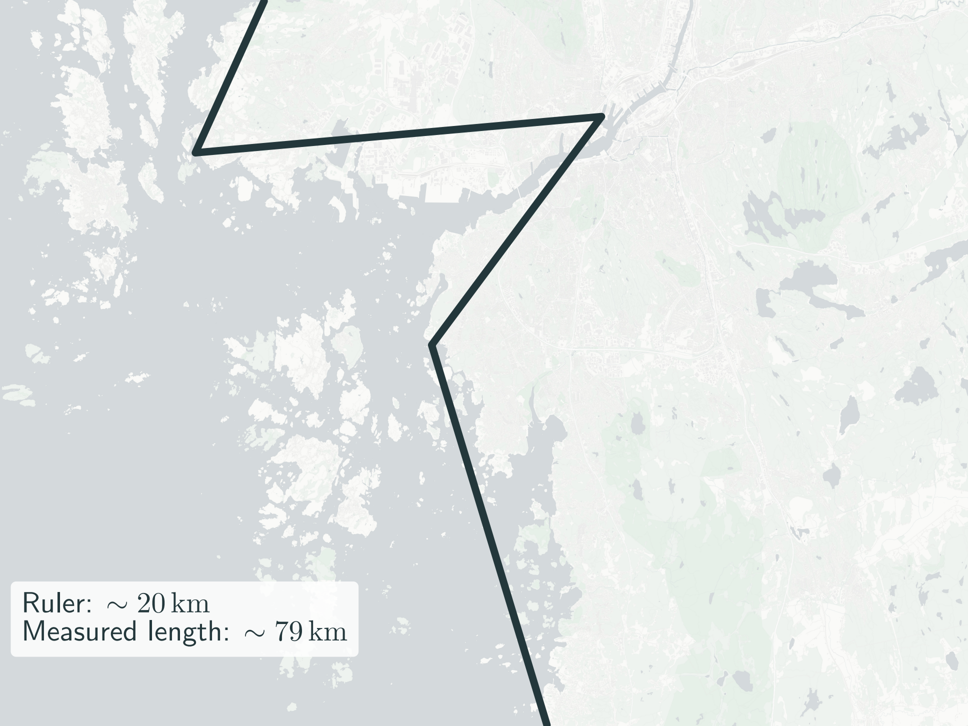
How to measure: a fractal paradox

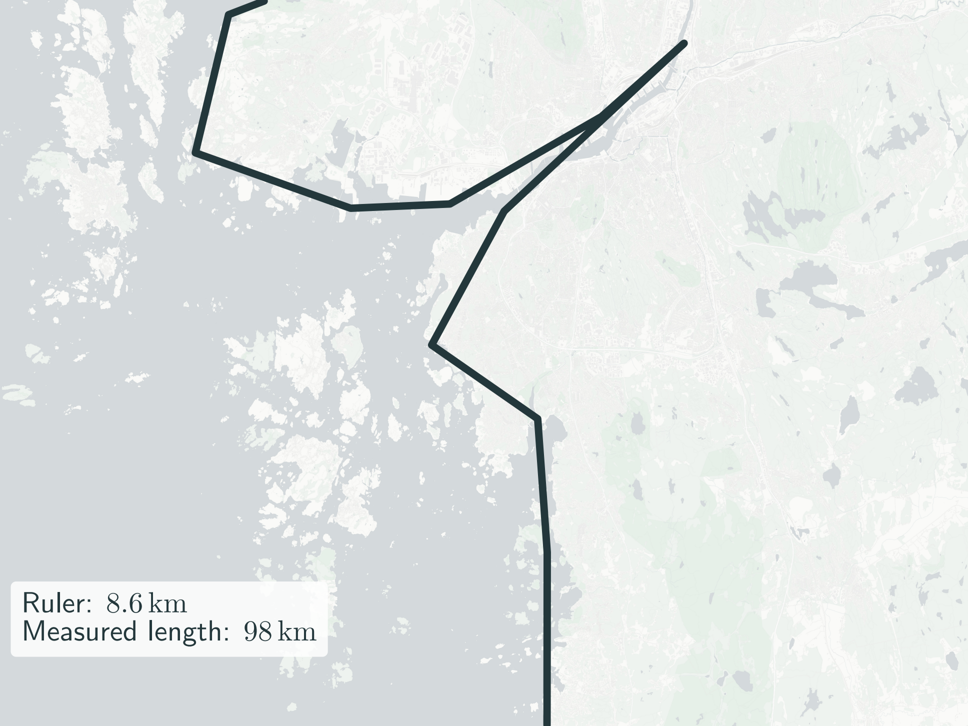
From coastlines to stochastic processes

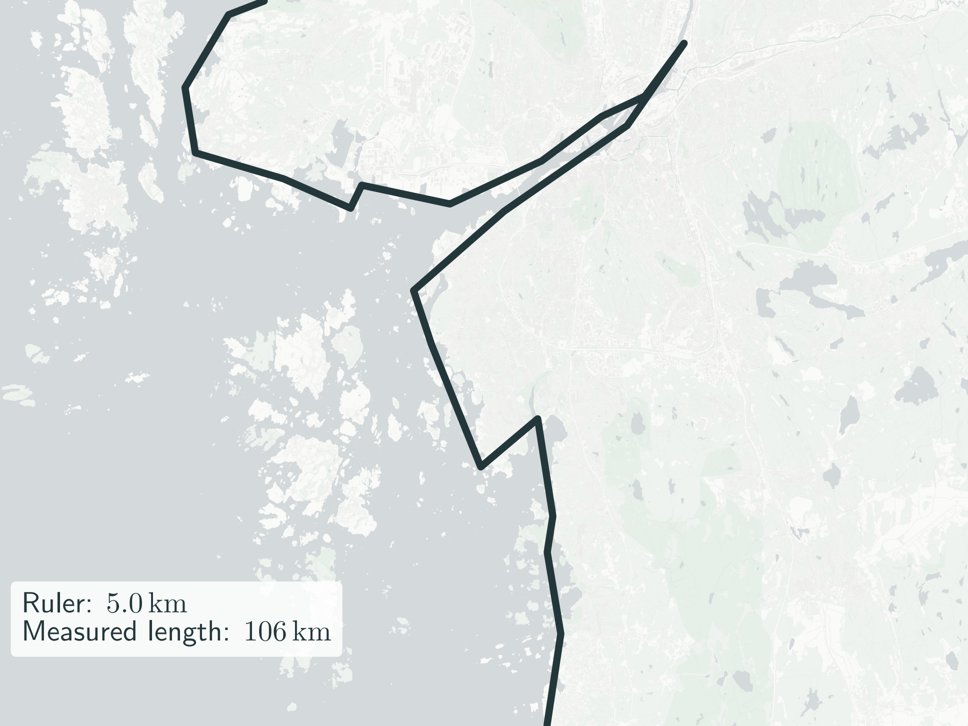
PhD Seminar, Kasper Bågmark

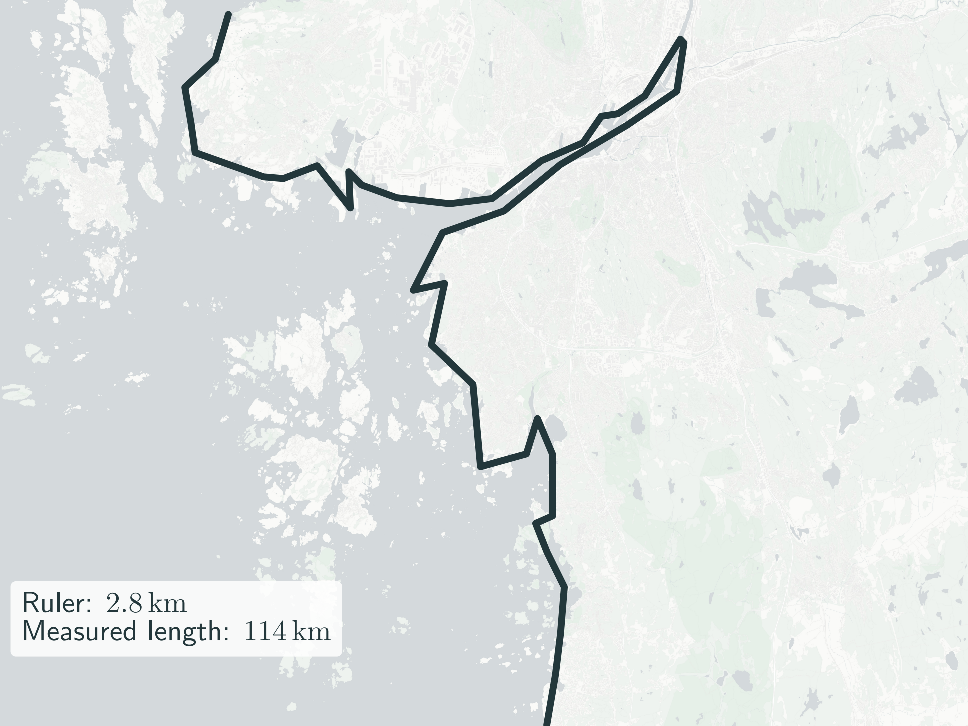
January, 2026

How long is the coastline?

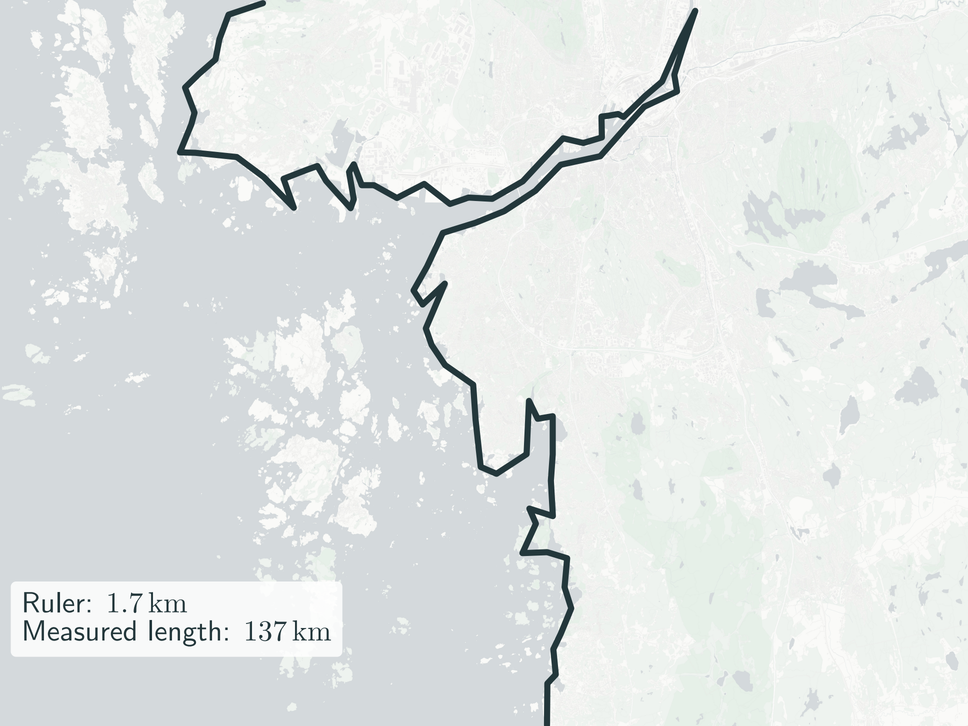

Ruler: $\sim 20\,\mathrm{km}$
Measured length: $\sim 79\,\mathrm{km}$

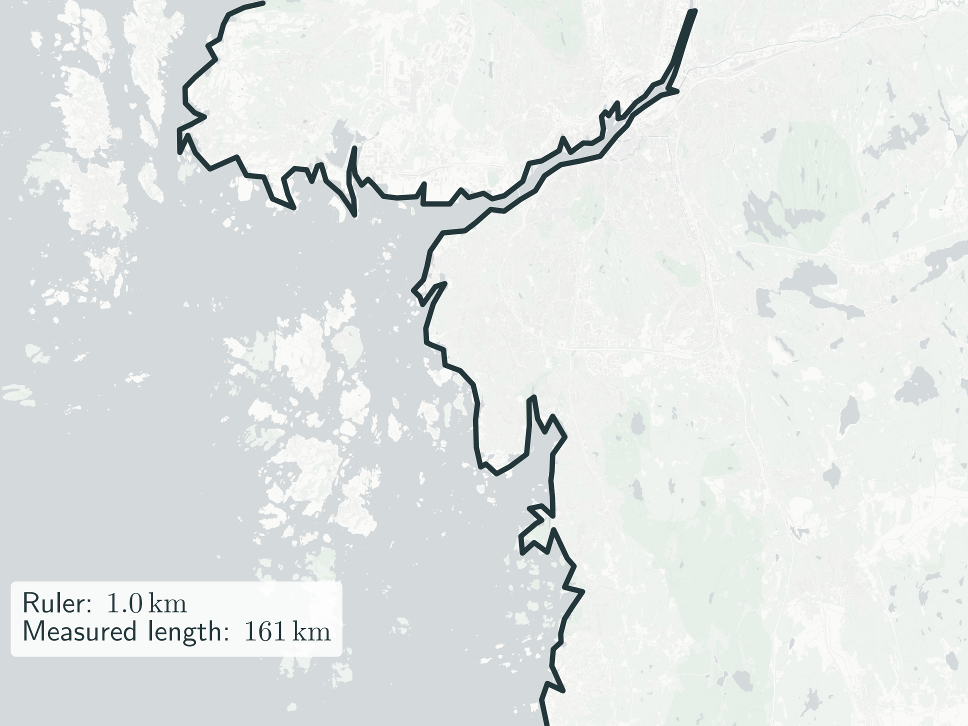

Ruler: 8.6 km
Measured length: 98 km

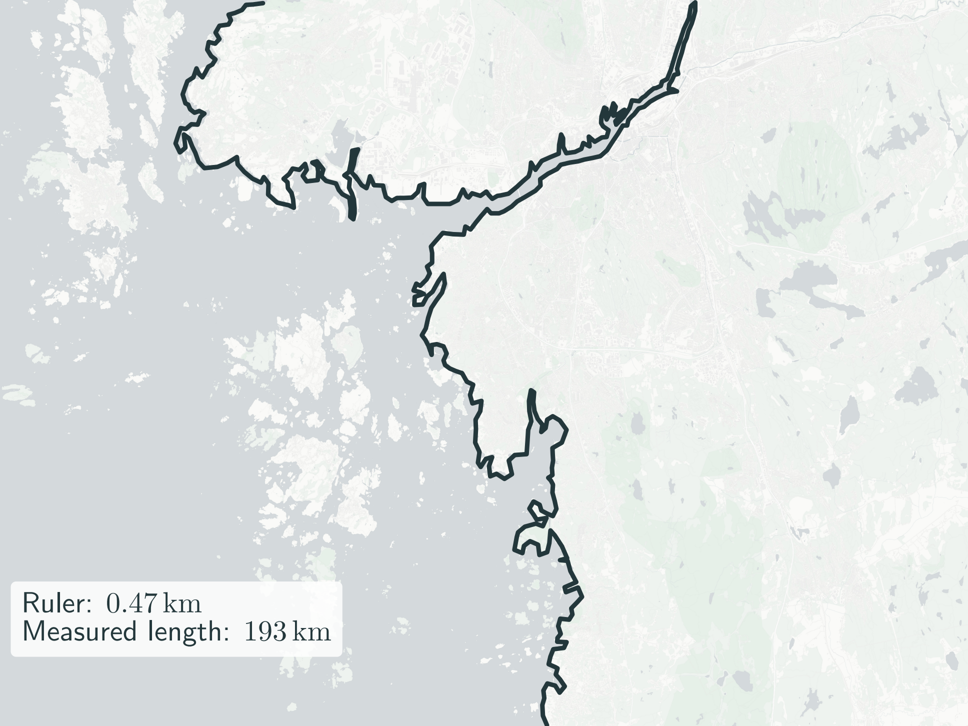

Ruler: 5.0 km
Measured length: 106 km

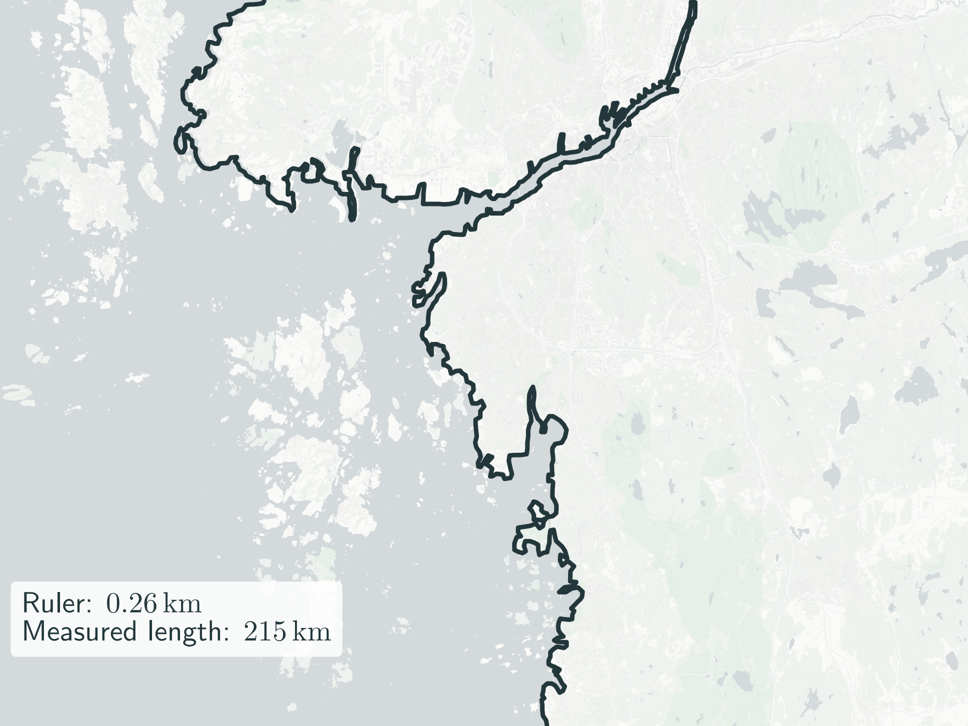


Ruler: 2.8 km
Measured length: 114 km


Ruler: 1.7 km
Measured length: 137 km


Ruler: 1.0 km
Measured length: 161 km


Ruler: $0.47\,\mathrm{km}$
Measured length: $193\,\mathrm{km}$

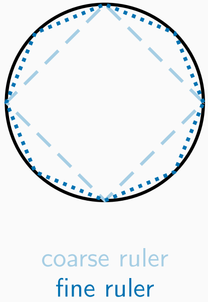

Ruler: 0.26 km
Measured length: 215 km

so what do we report?

Ruler: $0.26\,\mathrm{km}$
Measured length: $215\,\mathrm{km}$

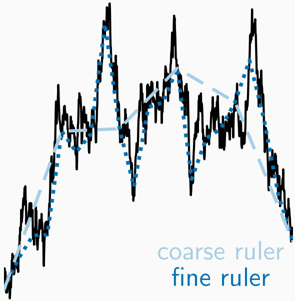
Measurement depends on scale

Smooth curve (circle)



Measured length converges as ruler ↓

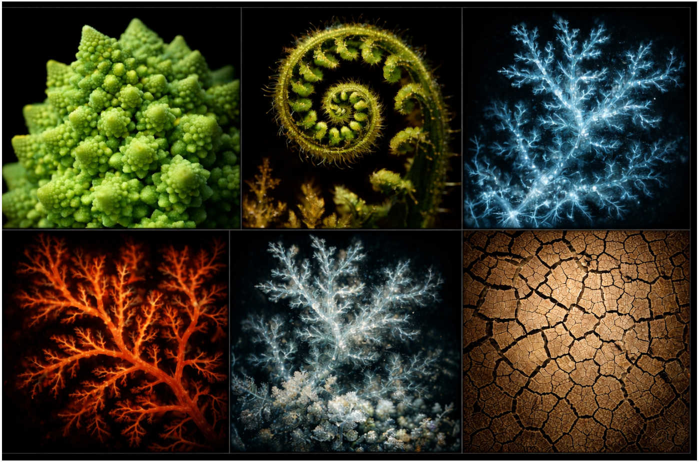
Rough curve (Brownian path)



Measured length grows as ruler ↓

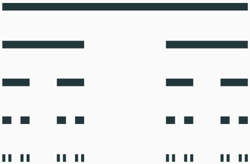
- Measuring length ≈ straight-line approximation
- Smooth curves ⇒ length converges
- Fractal paths ⇒ length does not converge

Fractals: structure at every scale

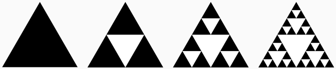


- "A Fractal is a rough or fragmented geometric shape that can be split into parts, each of which is (at least approximately) a reduced-size copy of the whole" - Benoit B. Mandelbrot

Fractals - self-similarity



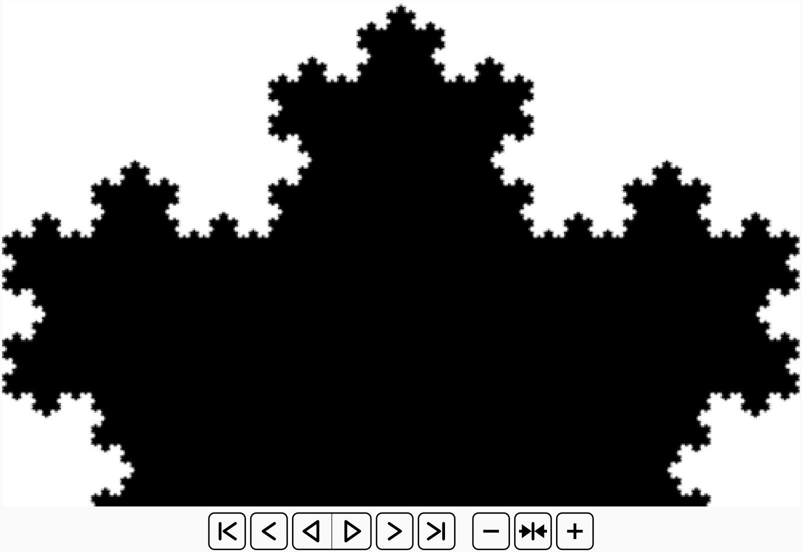
The Cantor-1/3 set



The Sierpinski triangle in $\mathbb{R}^2$

The von Koch Snowflake

Fractals - von Koch Snowflake

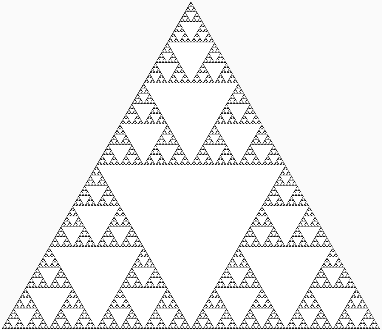


- Exact self-similarity: identical at every scale

Fractals - not exact self-similar

- What length should the ruler have to measure a Brownian motion?

Fractals - Measuring

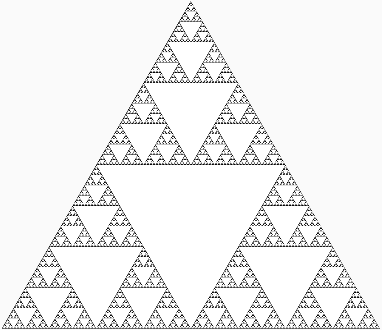


The 2-dimensional Sierpinski triangle

- Length (1-dimensional measure) of the curve? $\to \infty$
- Area (2-dimensional measure) of the set? $\to 0$
- We need a non-integer dimensional measure, this measure is the Hausdorff measure.

- Fractal dimension, or Hausdorff dimension, is a real $d \in \mathbb{R}$, which we can measure with respect to. For the Sierpinski triangle this should lie between 1 and 2.

Dimension from scaling



- Cover the set with boxes of size ε
- Let $N(\varepsilon)$ be the number of boxes needed
- Scaling law:

$$N(\varepsilon) \approx \varepsilon^{-d}$$

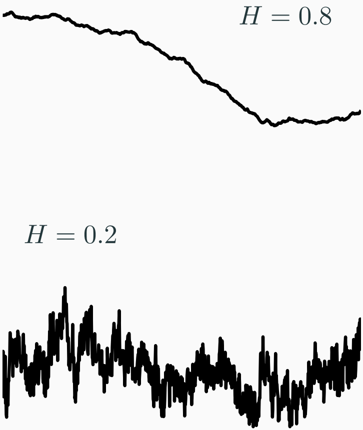
- The exponent d is the *fractal dimension*

Dimension measures how complexity grows under refinement

Brownian motion as a random fractal

- Continuous path
- Nowhere differentiable
- Scale-free roughness
- Hausdorff dimension $d = \frac{3}{2}$

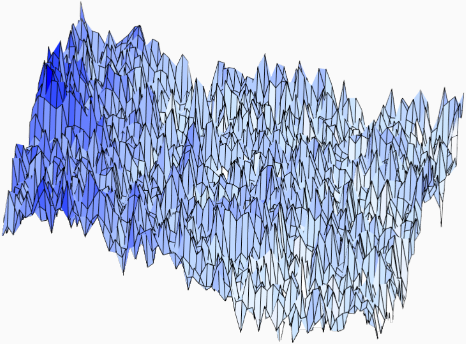
Fractional Brownian motion: tuning roughness



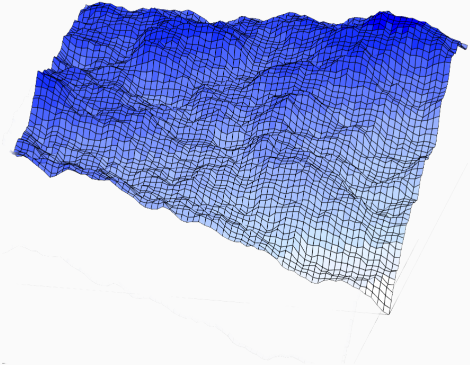
- Family of random fractal curves
- Controlled by the Hurst parameter $H \in (0, 1)$
- Small $H \Rightarrow$ very rough paths
- Large $H \Rightarrow$ smoother paths
- Classical Brownian motion $H = \frac{1}{2}$
- Hausdorff dimension $d = 2 - H$

Fractional Brownian field

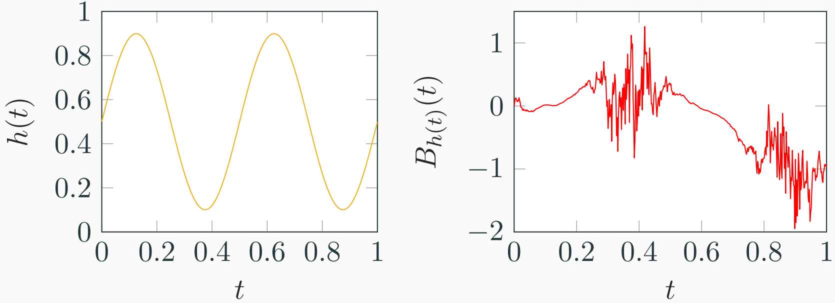
$H = 0.2$



$H = 0.8$

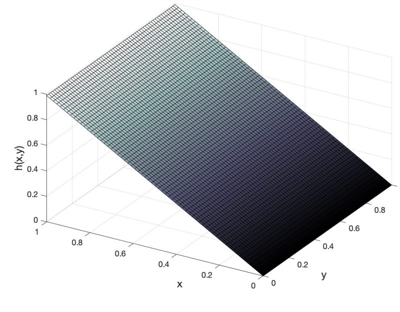


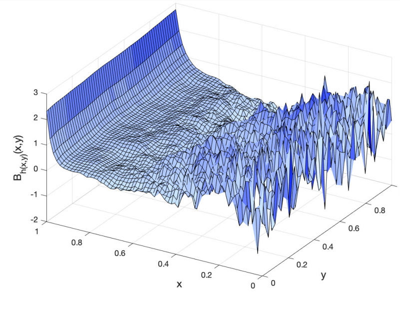
Multifractional Brownian motion



To the left: Hurst parameter H as $h(t)$. To the right: A corresponding sample path

Multifractional Brownian field





To the left: Hurst parameter as $h(x, y) = x$. To the right: A corresponding sample path

Conclusion

I am doing a PhD seminar at our department, can you calculate the Hausdorff dimension of the coastline of Gothenburg for me?

Thought for 2m 28s ›

You *can't* meaningfully "calculate the Hausdorff dimension of the coastline of Gothenburg" as a single, objective number without first fixing (i) **what you mean by "the coastline"** (mainland only? all islands in the archipelago? which administrative boundary?) and (ii) the **range of spatial scales** you regard as relevant. That's the whole punchline of the *coastline paradox*: measured geometric complexity changes with measurement scale, and natural coastlines aren't exact self-similar fractals. Wikipedia

What you *can* report in a seminar (and what people usually do)

Use an **estimated box-counting (Minkowski) dimension** over a stated scale range, which is a standard practical proxy for "fractal dimension" of geographic curves. fractalfoundation.... +1

A slide-friendly statement would be something like:

> **Estimated fractal (box-counting) dimension of the Gothenburg coastline (incl. archipelago, scales ~100 m–10 km):**
>
> $$D \approx 1.2\text{–}1.3,$$
>
> with the exact value depending strongly on definition + dataset resolution.