Revisiting nonlinear filtering through deep BSDE methods:

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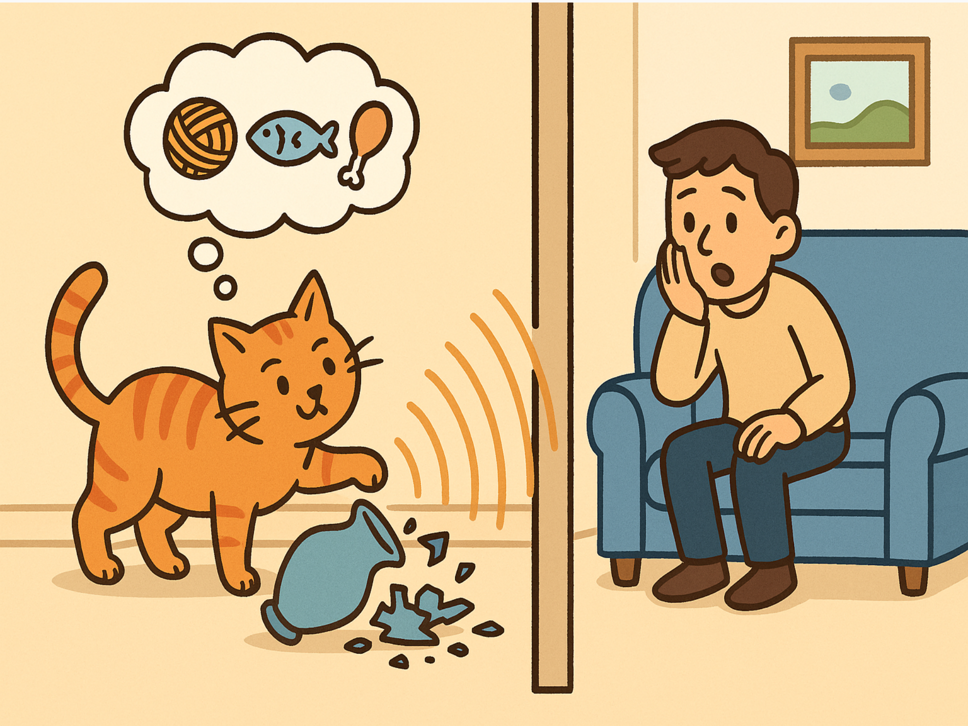
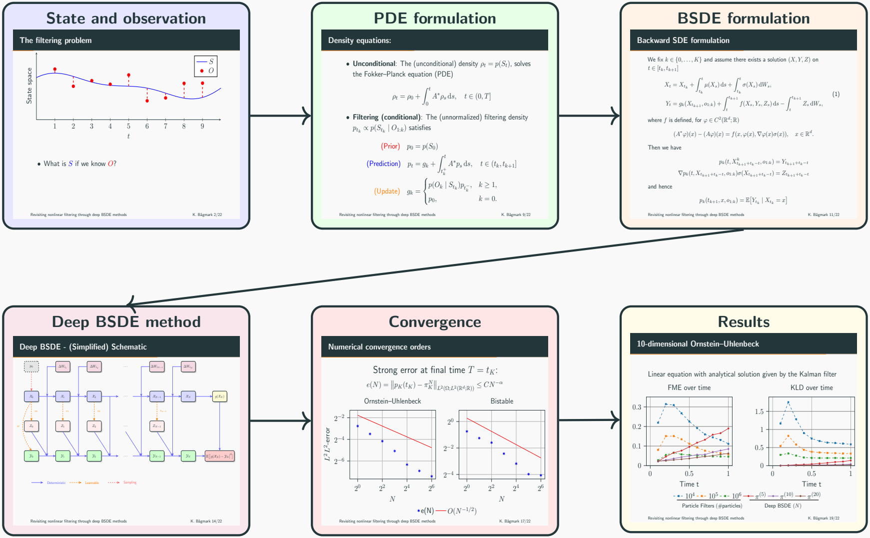


Summary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PDE formulation
Density equations:
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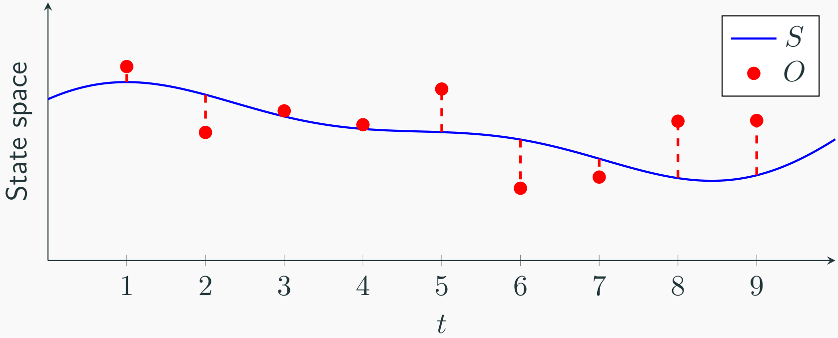
Convergence
Numerical convergence orders
Strong error at final time $T = t_K$:
Results
10-dimensional Ornstein–Uhlenbeck
Linear equation with analytical solution given by the Kalman filter

Introduction

The filtering problem



- Goal: Find the density p_{t_k}

$$\mathbb{P}(S_{t_k} \in B \mid O_{1:k}) = \int_B p_{t_k}(x \mid O_{1:k})\,\mathrm{d}x$$

Contributed method

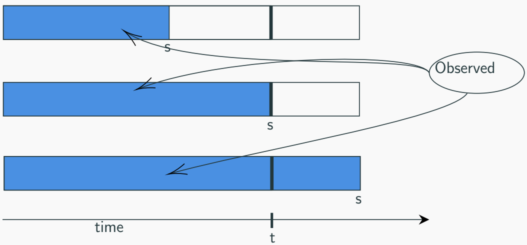
- Learns the map $O_{1:k} \to p(S_k \mid O_{1:k})$
  - Assumes known SDE coefficients and measurement model
- PDE based method, scalable in dimension of S
  - No spatial discretization required
  - Numerically stable for toy example in 100 dimensions
- Convergence
  - Strong convergence in the probabilistic representation
  - Convergence order empirically verified

Setting

The filtering problem

Goal: find $p(S_t \mid O_{0:s})$

- Prediction: $s < t$
- Filtering: $s = t$
- Smoothing: $s > t$



Continuous state

S solves the Stochastic Differential Equation (SDE)

$$S_t = S_0 + \int_0^t \mu(S_s)\,\mathrm{d}s + \int_0^t \sigma(S_s)\,\mathrm{d}B_s, \quad t \in [0, T].$$

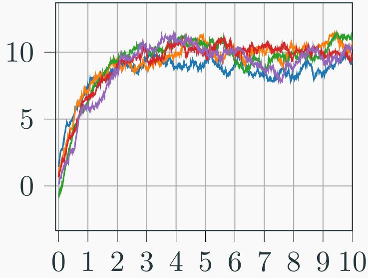
Infinitesimal generator A

$$A\varphi = \frac{1}{2}\sum_{i,j=1}^{d} a_{ij}\,\frac{\partial^2 \varphi}{\partial x_i \partial x_j} + \sum_{i=1}^{d} \mu_i\,\frac{\partial \varphi}{\partial x_i}$$

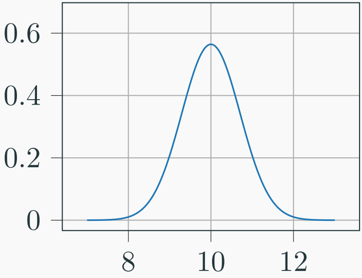
where $a := \sigma\sigma^\top$

Example: Ornstein–Uhlenbeck process (linear SDE)

$$S_t = S_0 - \int_0^t (S_s - 10)\,\mathrm{d}s + B_t, \quad t \in [0, T],$$
$$S_0 \sim p_0 = \mathcal{N}(0, 1)$$



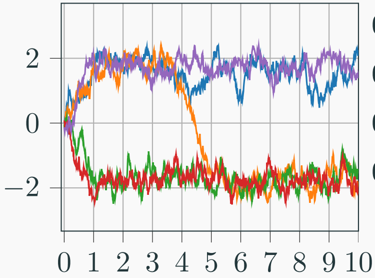
State $(S_t)_{t\in[0,T]}$



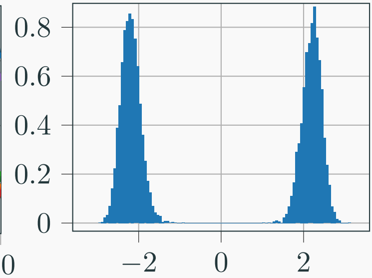
Distribution $p(S_T)$

Example: Bistable process (nonlinear SDE)

$$S_t = S_0 + \int_0^t (5S_s - S_s^3)\,\mathrm{d}s + B_t, \quad t \in [0,T],$$
$$S_0 \sim p_0 = \mathcal{N}(0,1)$$



State $(S_t)_{t\in[0,T]}$



Distribution $p(S_T)$

Discrete observations

- **Measurement model**:

$$O_k = h(S_{t_k}) + U_k, \quad k = 1, \ldots, K,$$

where $h\colon \mathbb{R}^d \to \mathbb{R}^{d'}$ and $U_k \sim \mathcal{N}(0, R_k)$.

- **Goal**: Find $p(S_{t_k} \mid O_{1:k})$ for $k = 1, \ldots, K$.

Classical methods

Linear and Gaussian case

Exact solutions:

- Prediction and filtering:
  The Kalman filter (Kalman–Bucy 1960)
- Smoothing:
  Rauch–Tung–Striebel smoother (1965)

Nonlinear case

Approximations:

- Approximative Kalman filter (Extended, Unscented, Ensemble)
- Particle filters (sequential Monte Carlo)

PDE formulation

Density equations:

- **Unconditional**: The (unconditional) density $\rho_t = p(S_t)$, solves the Fokker–Planck equation (PDE)

$$\rho_t = \rho_0 + \int_0^t A^* \rho_s \, \mathrm{d}s, \quad t \in (0, T]$$

- **Filtering (conditional)**: The (unnormalized) filtering density $p_{t_k} \propto p(S_{t_k} \mid O_{1:k})$ satisfies

$$\text{(Prior)} \quad p_0 = p(S_0)$$

$$\text{(Prediction)} \quad p_t = g_k + \int_{t_k^+}^t A^* p_s \, \mathrm{d}s, \quad t \in (t_k, t_{k+1}]$$

$$\text{(Update)} \quad g_k = \begin{cases} p(O_k \mid S_{t_k}) p_{t_k^-}, & k \geq 1, \\ p_0, & k = 0. \end{cases}$$

BSDE approach - brief outline

Backward SDE formulation

Define f, for $\varphi \in C^2(\mathbb{R}^d; \mathbb{R})$, by

$$(A^*\varphi)(x) - (A\varphi)(x) = f(x, \varphi(x), \sigma(x)^\top \nabla\varphi(x)), \quad x \in \mathbb{R}^d.$$

We fix $k \in \{0, \ldots, K\}$ and assume there exists a solution (X, Y, Z) on $t \in [t_k, t_{k+1}]$

$$\begin{aligned} X_t &= X_{t_k} + \int_{t_k}^{t} \mu(X_s)\,\mathrm{d}s + \int_{t_k}^{t} \sigma(X_s)\,\mathrm{d}W_s, \\ Y_t &= g_k(X_{t_{k+1}}, o_{1:k}) + \int_{t}^{t_{k+1}} f(X_s, Y_s, Z_s)\,\mathrm{d}s - \int_{t}^{t_{k+1}} Z_s^\top\,\mathrm{d}W_s. \end{aligned} \tag{1}$$

Then we have

$$p_k(t, X^k_{t_{k+1}+t_k-t}, o_{1:k}) = Y_{t_{k+1}+t_k-t} \tag{2}$$

$$\sigma(X_{t_{k+1}+t_k-t})^\top \nabla p_k(t, X_{t_{k+1}+t_k-t}, o_{1:k}) = Z_{t_{k+1}+t_k-t} \tag{3}$$

and hence

$$p_k(t_{k+1}, x, o_{1:k}) = \mathbb{E}\big[Y_{t_k} \mid X_{t_k} = x\big] \tag{4}$$

Continuous optimization formulation

The solution p is the solution to the following minimization problem

$$\min_{u \in C([t_k, t_{k+1}] \times \mathbb{R}^d \times \mathbb{R}^{d' \times k}; \mathbb{R})} \mathbb{E}\left[\left|Y_{t_{k+1}}^{(u,o_{1:k})} - g_k(X_{t_{k+1}}, o_{1:k})\right|^2\right]$$

$$Y_t^{(u,o_{1:k})} = u(t, X_t, o_{1:k}), \quad Z_t^{(u,o_{1:k})} = \sigma(X_t)^\top \nabla u(t, X_t, o_{1:k})$$

$$X_t = X_{t_k} + \int_{t_k}^t \mu(X_s)\,\mathrm{d}s + \int_{t_k}^t \sigma(X_s)\,\mathrm{d}W_s, \quad t \in [t_k, t_{k+1}],$$

$$Y_t^{(u,o_{1:k})} = Y_{t_k}^{(u,o_{1:k})} - \int_{t_k}^t f(X_s, Y_s^{(u,o_{1:k})}, Z_s^{(u,o_{1:k})})\,\mathrm{d}s + \int_{t_k}^t (Z_s^{(u,o_{1:k})})^\top\,\mathrm{d}W_s, \quad t \in [t_k, t_{k+1}].$$

The discrete optimization problem

- Define a finer time partition $t_k = t_{k,0} < t_{k,1} < \cdots < t_{k,N} = t_{k+1}$
- Approximate $(X_t, Y_t)_{t \in [t_k, t_{k+1}]}$ with Euler–Maruyama $(\mathcal{X}_n, \mathcal{Y}_n)_{n=0}^N$
- Approximate $u(t_k, x) \approx w(x)$ and $\nabla u(t_{k,n}, x) \approx v_n(x)$, $n = 0, \ldots, N-1$

$$\min_{\substack{w \in C(\mathbb{R}^d \times \mathbb{R}^{d' \times k}; \mathbb{R}) \\ (v_n)_{n=0}^{N-1} \in C(\mathbb{R}^d \times \mathbb{R}^{d' \times k}; \mathbb{R}^d)^N}} \mathbb{E}\Big[\Big|\mathcal{Y}_N^{o_{1:k}} - \bar{g}_k(\mathcal{X}_N, o_{1:k})\Big|^2\Big]$$

$$\mathcal{Y}_0^{o_{1:k}} = w(\mathcal{X}_0, o_{1:k}),$$

for $n = 0, \ldots, N-1$

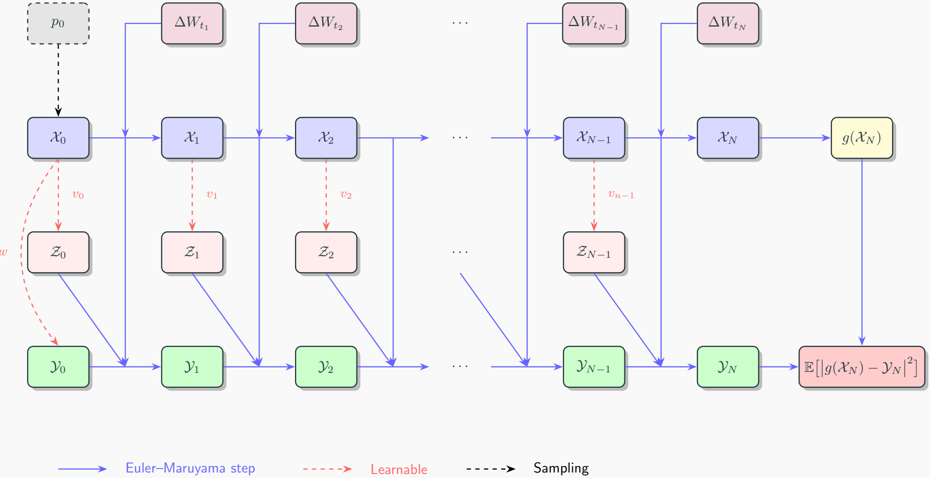
$$\mathcal{Z}_n^{o_{1:k}} = \sigma(\mathcal{X}_n)^\top v_n(\mathcal{X}_n, o_{1:k}),$$

$$\mathcal{X}_{n+1} = \mathcal{X}_n + \mu(\mathcal{X}_n)(t_{k,n+1} - t_{k,n}) + \sigma(\mathcal{X}_n)(W_{t_{k,n+1}} - W_{t_{k,n}}),$$

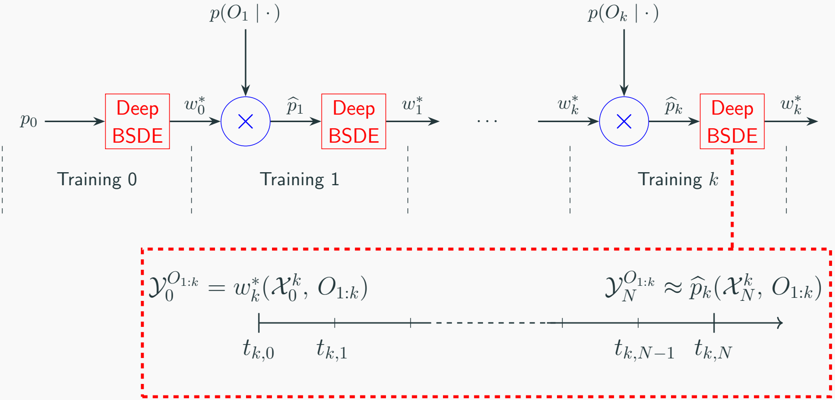
$$\mathcal{Y}_{n+1}^{o_{1:k}} = \mathcal{Y}_n^{o_{1:k}} - f\big(\mathcal{X}_n, \mathcal{Y}_n^{o_{1:k}}, \mathcal{Z}_n^{o_{1:k}}\big)(t_{k,n+1} - t_{k,n}) + (\mathcal{Z}_n^{o_{1:k}})^\top (W_{t_{k,n+1}} - W_{t_{k,n}}).$$

Let $\widehat{p}_k^N(x, o_{1:k}) = w^*(x, o_{1:k-1}) p(O_k = o_k \mid S_{t_k} = x)$ define our deep BSDE filter

Deep BSDE - (Simplified) Schematic



deep BSDE filter



Numerical convergence

Error bound

- p - true filter solution
- $(\mathcal{X}, \mathcal{Y}, \mathcal{Z})$ - deep BSDE approximation
- $\widehat{p}$ - filter approximation

Under sufficient conditions there exists a constant C such that, for all $k = 1, \ldots, K$

$$\left\|p_k(t_k) - \widehat{p}_k^N\right\|_{L^\infty(\mathbb{O};L^\infty(\mathbb{R}^d;\mathbb{R}))} \leq C\left(N^{-\frac{1}{2}} + \sum_{j=0}^{K-1} \sup_{o\in\mathbb{O}} \sup_{x\in\mathbb{R}^d} \mathbb{E}\left[\left|\overline{g}_j(\mathcal{X}_N^{j,x}, o_{1:j}) - \mathcal{Y}_N^{j,x}\right|^2\right]^{\frac{1}{2}}\right)$$

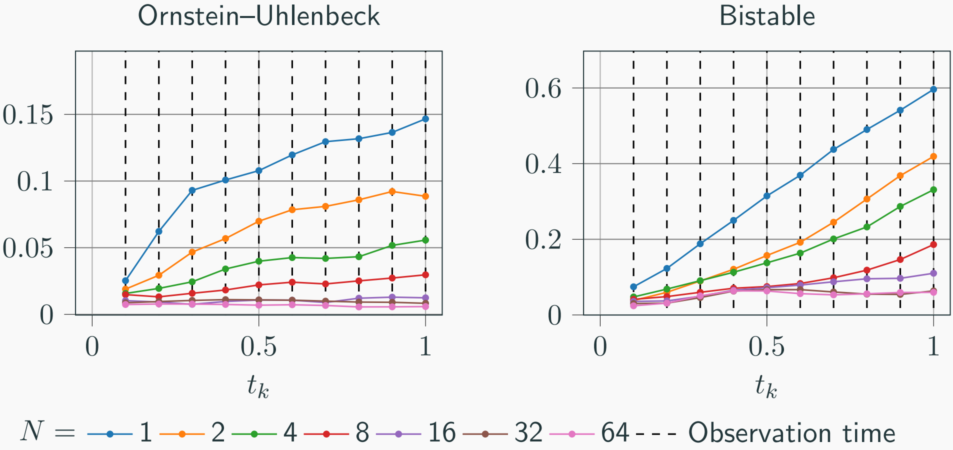
Define

$$e_k(N) = \left\|p_k(t_k) - \widehat{p}_k^N\right\|_{L^\infty(\mathbb{O};L^\infty(\mathbb{R}^d;\mathbb{R}))}$$

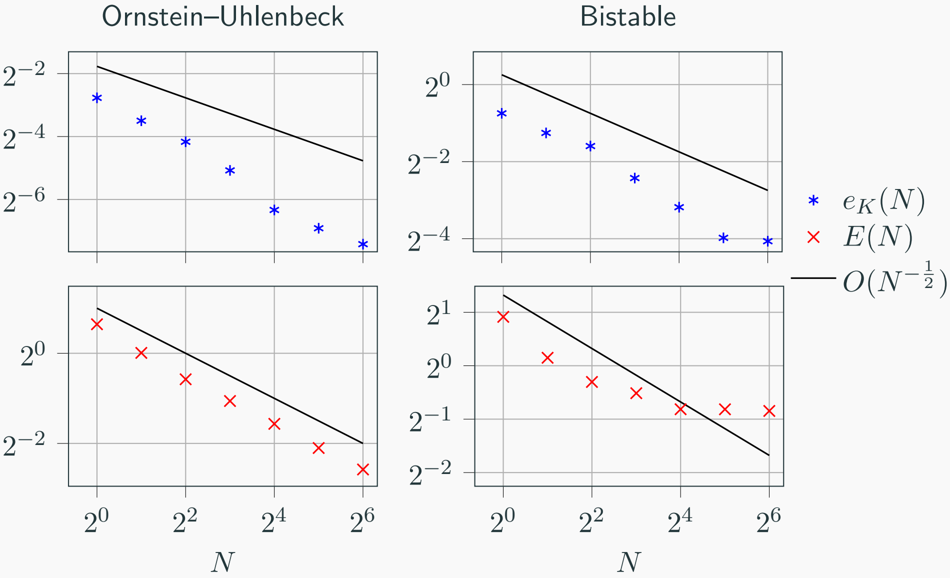
$$E(N) = \sum_{j=0}^{K-1} \sup_{o\in\mathbb{O}} \sup_{x\in\mathbb{R}^d} \mathbb{E}\left[\left|\overline{g}_j(\mathcal{X}_N^{j,x}, o_{1:j}) - \mathcal{Y}_N^{j,x}\right|^2\right]^{\frac{1}{2}}$$

Numerical convergence

Strong error $e_k(N) = \left\| p_k(t_k) - \widehat{p}_k^N \right\|_{L^\infty(\mathbb{O};L^\infty(\mathbb{R}^d;\mathbb{R}))}$ for $k = 1, \ldots, 10$



Numerical convergence orders



Comparison to particle filters

Metrics

Let μ and p denote the true mean and density, $\widehat{\mu}$ and $\widehat{p}$ a generic approximation

First Moment Error

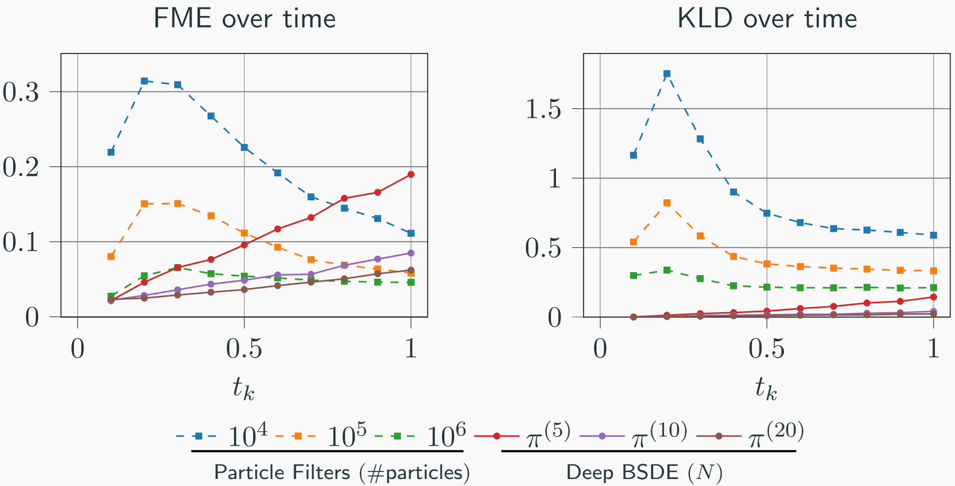
$$\mathsf{FME} = \frac{1}{M} \sum_{m=1}^{M} \left\| \mu_k^{(m)} - \widehat{\mu}_k^{(m)} \right\|, \quad \text{for } k = 1, \dots, K.$$

Forward averaged Kullback–Leibler Divergence

$$\mathsf{KLD} = \frac{1}{MN} \sum_{m=1}^{M} \sum_{n=1}^{N} \log \left(\frac{p_{t_k}^{(m)}\left(x^{(n,m)}\right)}{\widehat{p}_k^{(m)}\left(x^{(n,m)}\right)} \right), \;\; x^{(n,m)} \sim p_{t_k}^{(m)}, \;\; \text{for } k = 1, \dots, K$$

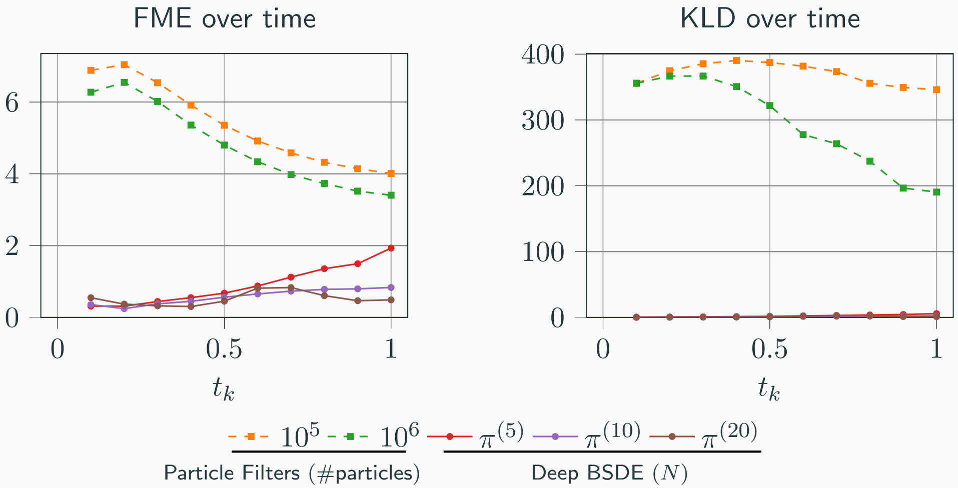
10-dimensional Ornstein–Uhlenbeck

Linear equation with analytical solution given by the Kalman filter



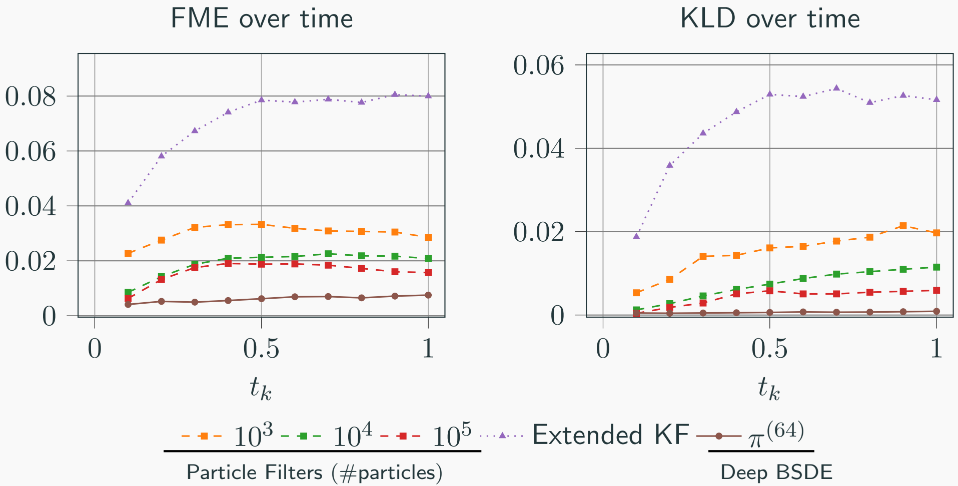
100-dimensional Ornstein–Uhlenbeck

Linear equation with analytical solution given by the Kalman filter



Bistable process

Nonlinear equation with a particle filter as reference solution



Summary

Goal: Solve the filtering problem without the curse of dimensionality

1. PDE formulation - the Fokker–Planck equation with updates
2. Reformulate as a BSDE
3. Euler–Maruyama and neural network approximation

Obtained objectives:

	Low-dimensional	**High-dimensional**
Linear	Particle Filter ✓ Proposed ✓	Particle Filter ✗ Proposed ✓
Nonlinear	Particle Filter ✓ Proposed ✓	Particle Filter ✗ Proposed (?)